# Extending Black-Litterman Analysis Beyond the Mean-Variance Framework

*An application to hedge fund style active allocation decisions.*

Lionel Martellini and Volker Ziemann

One by-product of the bull market of the 1990s has been establishment of hedge funds as an important segment of the financial markets. In 2004 the value of the hedge fund industry worldwide was estimated at over $1 trillion, with approximately 7,000 hedge funds in the world.

There seem to be two main reasons behind the success of hedge funds in institutional money management. On the one hand, it is argued that hedge funds may provide abnormal risk-adjusted returns, due to the superior skills of hedge fund managers and flexibility in trading strategies. On the other hand, hedge funds seem to provide diversification benefits with respect to other investment possibilities. Investors can take advantage of hedge funds' linear and non-linear exposure to a great variety of risk factors, including volatility and credit and liquidity risk, to reduce the risk of a global portfolio.

In an attempt to fully capitalize on such diversification benefits in a top-down approach, investors or funds of hedge funds managers must be able to rely on robust techniques for optimization of portfolios including hedge funds. Standard mean-variance portfolio selection techniques are known to suffer from a number of shortcomings, and a variety of problems are exacerbated in the presence of hedge funds. It has been argued that the two most important aspects that require specific care are the presence of non-normally distributed assets and parameter uncertainty.

First, because hedge fund returns are not normally distributed, a mean-variance optimization would be severely ill-adapted, except for an investor with quadratic preferences. That mixing hedge funds with traditional assets leads to reduced volatility in a traditional portfolio

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with a constant return level has been noted by many (e.g., Terhaar, Staub, and Singer [2003]), owing to the fact that hedge funds (at least for some strategies) present low volatility together with low correlation with traditional asset classes. It can be shown at the same time, through a statistical model integrating fatter tails than those of the normal distribution, that minimizing the second-order moment (the volatility) can be accompanied by a significant increase in extreme risks (Sornette, Anderson, and Simonetti [2000]). This is confirmed by Amin and Kat [2003], who find empirical evidence that low volatility is generally obtained at the cost of lower skewness and higher kurtosis. Consequently, according to Cremers, Kritzman, and Page [2005], in the presence of asymmetric or fat-tailed return distribution functions, the use of mean-variance analysis can potentially lead to a significant loss of utility for investors.

Many attempts have been made to better account for the specific risk features of hedge funds and to extend portfolio optimization techniques to account for the presence of fat-tailed distributions, mostly by introducing some risk objective beyond volatility (e.g., value at risk (VaR) as in Favre and Galeano [2002], *conditional value at risk* as in Agarwal and Naik [2004] and DeSouza and Gokcan [2004] or the *omega ratio* as in Favre-Bulle and Pache [2003]), integrating the presence of non-trivial higher moments in asset returns.

Second, the problem of parameter uncertainty needs to be addressed, as the lack of a long history and the non-availability of high-frequency data imply that parameter estimation is a real challenge in the case of hedge fund returns. There are many reasons to believe that the main challenge for optimal allocation models is in fact in estimating expected returns as opposed to higher moments of asset class returns.

There is a general consensus that expected returns are difficult to obtain with a reasonable estimation error, a problem that is all the more acute in the hedge fund universe where data are scarce and there are a number of performance biases (see, for example, Fung and Hsieh [2000, 2002]). It makes the problem worse that optimization techniques are very sensitive to differences in expected returns, so that portfolio optimizers typically allocate the greatest propotion of capital to the asset class for which estimation error in the expected returns is the highest (e.g., Michaud [1998] or Britten-Jones [1999]).

Significant progress on the question of portfolio optimization in the presence of parameter uncertainty has been achieved in influential work by Black and Litterman [1990, 1992], who introduce, in a static mean-variance setting, a methodology that allows investors to account for uncertainty in their priors on expected returns, expressed in terms of deviation from neutral equilibrium CAPM-based estimates.[1] While the Black-Litterman approach is appealing because of its technical tractability and conceptual simplicity, and it has become widely used in active asset allocation decisions, it is not well suited for hedge fund portfolio optimization because of its focus on normally distributed assets.

What is still missing for active style allocation in the hedge fund universe is a model that would take both non-normal distribution and parameter uncertainty into account. Our contribution is an optimal allocation model that incorporates an answer to both challenges within a unified framework. To this end, we first introduce a suitable extension of the Black-Litterman Bayesian approach to portfolio construction that allows for the incorporation of active views about hedge fund strategy performance in the presence of non-trivial preferences about higher moments of hedge fund return distributions.

Our work is closely related to Giacometti et al. [2005], who extend the Black-Litterman model to a more general class of return distributions, the stable distributions, and solve the allocation problem for various risk measures including value at risk and conditional value at risk. We differ from them by using a non-parametric approach that can be used for general return distributions.

Also related is Meucci [2006], who introduces a procedure for extending the Black-Litterman framework to generic market distributions and shows an application to portfolio management in a thick-tailed environment. This extension of the Black-Litterman model, however, comes at the cost of analytical results, and numerical methods have to be used to solve the problem. Because we focus on an approximation of the joint distribution of asset returns in terms of third- and fourth-order moments and co-moments (co-skewness and co-kurtosis), as opposed to the distribution itself, we are able to maintain analytical results.

We also present a numerical application illustrating how investors can use a multifactor approach to generate such active views and dynamically adjust their allocation to various hedge fund strategies while staying coherent with a long-term strategic allocation benchmark. The bulk of our message is straightforward. It is only by taking into account the exact nature and composition of an investor's current portfolio, as opposed to considering hedge fund investing on a stand-alone basis, that institutional investors can truly customize a portfolio and maximize the benefits they can expect from investing in these modern forms of alternative investment strategies.

Our results overall strongly suggest that significant value can be added in a hedge fund portfolio through the systematic implementation of active style allocation decisions, at both the strategic and tactical levels.

## A BAYESIAN MODEL OF ACTIVE HEDGE FUND STYLE ALLOCATION DECISIONS

We first present a formal model of active style allocation that can be used in the hedge fund universe. We review the Black-Litterman model, and then extend it to take into account higher moments of the return distribution.

### Review of the Black-Litterman Approach

Following the seminal work of Markowitz [1952], there is a strong consensus in portfolio management on the trade-off between expected return and risk. In the Markowitz world, risk is represented by the standard deviation. Given an investor's specific risk aversion, optimal portfolios and the so-called efficient frontier can be derived. Sharpe [1964] and Lintner [1965] use this mean-variance approach to design an equilibrium model, the capital asset pricing model (CAPM), aimed at describing asset returns. Assuming homogeneous beliefs, every investor holds the market portfolio derived from this equilibrium model.

Since then, Black and Litterman [1990, 1992] have proposed a formal model based on the desire to combine neutral views consistent with market equilibrium and individual active views. They introduce confidence levels on the prior distribution and on individual beliefs and obtain the joint distribution. Based on a Bayesian approach, the expected return incorporates market views and individual expectations. The Black-Litterman approach in its original form can be summarized in the multi-step process below (see Idzorek [2004]).

Given the risk aversion coefficient ($\lambda$), the historical variance covariance matrix ($\Sigma$), and the vector of market capitalization weights ($w_M$), the vector of implied equilibrium returns in excess of the risk-free rate can be obtained as: $\Pi = \lambda \Sigma w_M$. This, of course, is equivalent to using a standard CAPM.[2]

The parameter $\lambda$ can thus be rewritten as:

$$\lambda = \left( \frac{E(R_M) - R_F}{\sigma_M^2} \right)$$

where the indexes $F$ and $M$ denote the risk-free rate and the market portfolio. Individual active views, perhaps based on forecasting procedures, are then introduced. These $k$ views can either be relative or absolute and are represented in the $k \times 1$ vector $Q$. The $k \times n$ projection matrix $P$ will be used to define these views: $Q = P \cdot R_A$.

A confidence level is associated with each of the views expressed in Q. Thus, the individual beliefs can be described by a normal view distribution: $P \cdot R_A \sim N(Q, \Omega)$. In the same way, the confidence in the equilibrium model and the derived implied returns can be defined. Consequently, we obtain the prior equilibrium distribution: $R_N \sim N(\Pi, \tau\Sigma)$.

Following the Bayesian rule, the two distributions are combined to yield the distribution (see Satchell and Scowcroft [2000]):

$$R_{BL} \sim N(E(R_{BL}), [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1})$$

where

$$E(R_{BL}) = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1} \times [(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q] \quad (1)$$

This distribution incorporates both the neutral (equilibrium) and the active views. Taking this expected return as an input, the optimal Black-Litterman portfolio weights ($w_{BL}$) are then given by:

$$w_{BL} = (\lambda\Sigma)^{-1} E(R_{BL})$$

### Extension to Higher Moments

While the Black-Litterman model is well-suited for portfolio construction in the context of active asset allocation decisions, it suffers from an important limitation, namely, that it is based on the Markowitz model, where volatility is used as the definition of risk. To adapt this approach to hedge fund portfolio construction, we propose a simple extension of the original Black-Litterman model that relies on the use of the four-moment capital asset pricing model, as opposed to the standard CAPM, in the estimation of the market-neutral implied views.

The four-moment CAPM is a straightforward extension of the standard CAPM for investors who have non-trivial preferences over higher-order moments of asset returns, which gives the expression for expected returns $\mu$ in excess of the risk-free rate $R_0$ (see, for example, Hwang and Satchell [1999]):

$$\mu - R_0 = \alpha_1 \beta^{(2)} + \alpha_2 \beta^{(3)} + \alpha_3 \beta^{(4)} \tag{2}$$

where $\beta^{(i)}$, the vectors of portfolio betas, are defined as:

$$\beta^{(2)} = \frac{\Sigma w}{\mu^{(2)}(R_p)}$$
$$\beta^{(3)} = \frac{\Omega_w}{\mu^{(3)}(R_p)}$$
$$\beta^{(4)} = \frac{\Psi_w}{\mu^{(4)}(R_p)}$$

where $\Omega_w$ (respectively, $\Psi_w$) is the vector of co-skewness parameters (respectively, co-kurtosis parameters) for the weight vector $w$ (see the appendix for more details). The variables $\alpha_i$ can be understood as the risk premiums associated with covariance, co-skewness, and co-kurtosis.

One way to obtain the $\hat{\alpha}_i$ estimates is in a simple generalized least squares regression. As our model will be applied to hedge funds, for which data are not available more often than monthly, however, we are constrained to use small samples. In this case, one is naturally led to mitigate sample risk through the introduction of a specific model, which of course comes at the cost of specification risk.

As in Jondeau and Rockinger [2006], we assume a specific representative utility function. We choose the constant absolute risk aversion function $U(W) = -e^{(-\lambda W)}$ for its appealing parameter interpretation. The risk premiums are then given by:

$$\alpha_1 = \frac{\lambda \mu^{(2)}(R_p)}{A}$$
$$\alpha_2 = -\frac{\lambda^2 \mu^{(3)}(R_p)}{2A}$$
$$\alpha_3 = \frac{\lambda^3 \mu^{(4)}(R_p)}{6A}$$

with

$$A = 1 + \frac{\lambda^2}{2}\mu^{(2)}(R_p) - \frac{\lambda^3}{6}\mu^{(3)}(R_p) + \frac{\lambda^4}{24}\mu^4(R_p)$$

The risk aversion parameter $\lambda$ can be calibrated with respect to historical data. Here we use long-term estimates based on stock return estimates over the 1900-2000 period for 16 countries by Dimson, Marsh, and Staunton [2002] to obtain $\lambda = 2.14$, based on a 6% risk premium and 17% volatility. The various co-moments are estimated over the whole sample period January 1997 through December 2004.

It is important to note that the vectors of portfolio betas as well as the risk premiums ($\alpha_i$) are functions in the weight vector $w$. Thus, Equation (2) gives us a deterministic relation between the expected returns (left-hand side) and the weighting vector (right-hand side). This equation can be used in two different ways: 1) to obtain implied expected return estimates based on exogenous portfolio weights, or 2) to obtain the weight vector based on exogenous expected returns.

We create a strategic style allocation benchmark that will define the "neutral" portfolio weights based on a minimization of the portfolio value at risk. Using Equation (2), those views will then be taken in order to obtain "neutral" expected returns, approach 1. We then introduce a prediction model that will be used to generate active views (expressed in terms of returns). The latter is incorporated with the neutral views applying the known Bayesian approach [Equation (1)]. Finally, Equation (2) allows us to obtain the resulting Black-Litterman portfolio weights, approach 2.

## DESIGNING A STRATEGIC STYLE ALLOCATION BENCHMARK

The key to the implementation of a structured top-down approach to investing in hedge funds is to properly define a target asset allocation across various hedge fund styles. A variety of investment constraints and objectives need to be taken into account in the design of the target allocation. In particular, the target allocation should be designed so as to allow for an optimal mix with the client's current stock and bond portfolio.

In the real world, investors are interested not only in maximizing expected return and minimizing volatility, but also in limiting the loss with a given probability $(1 - \alpha)$.

This is the reason our objective focuses on a measure like VaR at $(1-\alpha)\%$, which is defined as the negative value of the $\alpha$-quantile of the underlying return distribution. Assuming a normal distribution, this measure is simply given by: $VaR(1-\alpha)=-(\mu+z_\alpha\sigma)$, where $z_\alpha$ is the $\alpha$-quantile of the standard normal distribution.

Our method for optimal allocation decisions to hedge funds can be used in the absence of specific views on hedge fund strategy returns when higher moments of hedge fund return distribution need to be taken into account. We focus on the most widely used set of hedge fund strategies, long/short equity, event-driven, relative value, and CTA.[3] We use the S&P 500 index to proxy for equity market returns, a portfolio of 20% U.S. government bonds, 45% mortgage bonds, and 35% corporate bonds to proxy for bond market returns (using Lehman bond indexes), and the Edhec Alternative Indexes as proxies for the return on the selected hedge fund strategies.[4]

Exhibit 1 shows that strategies such as relative value or event-driven present low levels of volatility but significantly negative skewness and significantly positive excess kurtosis. This is the reason we should incorporate higher moments in the VaR measure. We can do this through a method called a Cornish-Fisher expansion, which represents a convenient approximation of distribution percentiles in the presence of non-Gaussian higher moments (see Jaschke [2002] for a detailed description).

The Cornish-Fisher expansion is derived from the general Gram-Charlier expansion, using the standard normal distribution as the reference function. For a four-moment approximation of $\alpha$-percentiles the formula is:

$$\tilde{z}_\alpha = z_\alpha + \frac{1}{6}(z_\alpha^2-1)S + \frac{1}{24}(z_\alpha^3-3z_\alpha)K - \frac{1}{36}(2z_\alpha^3-5z_\alpha)S^2$$

where $\tilde{z}_\alpha$ denotes the modified $\alpha$-percentile, $z_\alpha$ denotes the $\alpha$-percentile of the standard normal distribution, $S$ denotes the portfolio skewness, and $K$ denotes the portfolio excess kurtosis. We then obtain the modified VaR measure with confidence $(1-\alpha)$: $VaR_{mod}(1-\alpha) = -(\mu+\tilde{z}_\alpha\sigma)$, where $\sigma$ ($\mu$) denotes the portfolio standard deviation (expected return).[5]

We also impose the portfolio constraint that not less than 5% nor more than 40% is allocated to a single hedge fund strategy. Jagannathan and Ma [2003] actually argue that portfolio constraints, in addition to avoiding corner solutions in optimization techniques, allow one to achieve a better trade-off between specification error and sampling error similar to what can be achieved by statistical shrinkage techniques (see Jorion [1986] and Ledoit [2003]).

## EXHIBIT 1
**Analysis of Return Distribution of Monthly Asset Returns (January 1997-December 2004)**

| | Return | Volatility | Skewness | Exc. Kurtosis |
|---|---|---|---|---|
| Long/Short Equity | 0.96% | 2.14% | 0.07 | 0.99 |
| CTA Global | 0.74% | 2.70% | 0.10 | -0.16 |
| Relative Value | 0.81% | 0.99% | -1.25 | 3.55 |
| Event Driven | 0.93% | 1.71% | -2.08 | 10.49 |
| Proxy Stock Markets | 0.63% | 4.86% | -0.50 | 0.04 |
| Proxy Bond Markets | 0.27% | 1.03% | -0.79 | 1.41 |

We rebalance the optimally designed portfolio every three months, using a 36-month rolling window analysis in the optimization procedure. We further assume that the investor is not willing to allocate more than 15% to hedge funds. As the history of the Edhec Alternative Indexes starts in January 1997, the first calibration period is January 1997-December 1999. We therefore obtain out-of-sample returns for the optimally designed portfolios from January 2000 onward.

We use this portfolio as a strategic benchmark from which neutral expected return estimates are obtained.

## GENERATING ACTIVE VIEWS ON HEDGE FUND STRATEGIES AND APPLICATION TO ACTIVE STYLE ALLOCATION DECISIONS

Investors and fund of hedge funds managers typically adjust their allocation to various hedge fund strategies in response to changes in their expectations about expected returns to these strategies. The extended Black-Litterman model can be used to add the benefits of active style rotation decisions to the benefits of strategic allocation decisions.

There is now a consensus in empirical finance that expected asset returns as well as variances and covariances are to some extent predictable. Pioneering work on the predictability of asset class returns in the U.S. market was carried out by Keim and Stambaugh [1986], Campbell [1987], Campbell and Shiller [1988], Fama and French [1989], and Ferson and Harvey [1991]. Then some authors started to investigate this phenomenon on an international basis by studying the predictability of asset class returns in various national markets (see, for example, Bekaert and Hodrick [1992], Ferson and Harvey [1993, 1995], Harvey [1995], and Harasty and Roulet [2000]).

While there has been a significant amount of research on the predictability of traditional asset classes, and implications in terms of tactical asset allocation strategies, until recently very little was known about the predictability of returns in alternative vehicles such as hedge funds. Amenc, El Bied, and Martellini [2003] examine (lagged) multifactor models for the return on nine hedge fund indexes. The factors are chosen to measure the many dimensions of financial risks: market risks (proxied by stock prices, interest rates, and commodity prices), volatility risks (proxied by implicit volatilities from option prices), default risks (proxied by default spreads), and liquidity risks (proxied by trading volume).

This research shows that a parsimonious set of models capture a very significant amount of predictability for most hedge fund styles. The benefits in terms of tactical style allocation portfolios are potentially great. Even more spectacular results are obtained both for an equity-oriented portfolio mixing traditional and alternative investment vehicles and for a fixed-income-oriented portfolio mixing traditional and alternative investment vehicles.

The conclusion from this research is that there is at least as much evidence of predictability in hedge fund returns as there is in stock and bond returns. We present the results of a simple experiment showing how predictability in hedge fund returns can be exploited in an active asset allocation process. To alleviate the risk of model specification, we perform a simple in-sample analysis using the historical relationship between a set of factors and hedge fund returns. We want not to emphasize a specific econometric process that could generate signals about hedge fund style returns, but rather that there are sophisticated and reliable active asset allocation models available to investors or managers who have active views on the performance of hedge fund strategies.

### Forming Active Views on Hedge Fund Returns

In order to detect individual active views on various hedge fund styles, for each hedge fund strategy we obtain a bullish, a bearish, or a neutral view concerning the expected return based on a lagged factor analysis. The predictive factors are:

- Implied Volatility (VIX)—CBOE SPX Volatility VIX.
- First differences of the implied volatility.
- Commodity Index—Goldman Sachs.
- Term Spread—Lehman U.S. Treasury 5-7 years minus Lehman U.S. Treasury 1-3 years.
- Credit Spread—Lehman U.S. Universal: High Yield Corp. Red Yield minus Lehman U.S. Treasury 1-3 years.
- Value versus Growth—S&P 500 Barra/Value minus S&P 500 Barra/Growth.
- Small-Cap versus Large-Cap—S&P 600 Small Cap minus S&P 500 Composite.
- S&P 500 Composite return.
- T-Bill—Merrill Lynch T-bill 3 months.
- U.S. Dollar—US MAJOR CURRENCY MAR73 = 100 (FED) EXCHANGE INDEX.
- Bond return volatility—Calculated over one-month Lehman U.S. Aggregate returns.

The approach is based on a simple univariate conditional factor analysis. The idea is to consider each factor separately and to estimate the correlation between the hedge fund returns and the three-month lagged factor values. Over a sample period from October 1996 through September 2004, the values for each factor are classified into three different states of nature according to the corresponding distribution: low (0%-33%), medium (over 33%-66%), and high (over 66%-100%). Next, we consider the hedge fund returns for each strategy, conditional on the state of nature of the three-month lagged factor values. A letter [H (for high), M (for medium), or L (for low)] is associated with each combination, as a function of the difference between the conditional and the unconditional mean:[6]

- H indicates that the conditional mean return on the strategy is significantly greater than the unconditional mean return, suggesting a bullish view according to the factor.[7]
- L indicates that the conditional mean is significantly lower than the unconditional mean, suggesting a bearish view according to the factor.
- M indicates that there is no significant difference between conditional and unconditional mean.

## EXHIBIT 2

**Conditional Factor Analysis of Hedge Fund Returns**

| | Implied Volatility | | | Diff. Implied Volatility | | | Commodity Index | | | Term Spread | | | Credit Spread | | | Value minus Growth | | | Small minus Large | | | S&P 500 | | | T-Bill | | | US Dollar | | | Lehman Bond Return | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high | low | medium | high |
| Long/Short Equity | M | M | M | L | M | M | M | M | M | M | H | L | M | H | L | H | M | L | H | M | M | L | H | M | M | M | M | M | M | M | M | M | M |
| CTA Global | H | M | L | M | M | M | M | M | M | M | M | M | M | M | M | M | M | M | H | L | M | M | M | M | M | M | M | M | H | M | M | M | L |
| Relative Value | L | M | M | L | M | M | M | M | M | M | H | L | M | H | L | H | M | M | M | M | M | L | M | M | L | M | H | M | M | M | M | M | M |
| Event Driven | L | M | M | M | M | H | M | M | M | M | H | M | M | H | M | H | M | M | M | M | M | L | H | M | M | M | M | M | M | M | M | M | M |

We report the results of this analysis in Exhibit 2. Note that when value and size spreads are tight, when markets are posting average returns, or when term and credit spreads are narrow, long/short equity funds tend to achieve high returns three months later. This is consistent with the fact that these funds typically hold—for liquidity reasons—long positions in small companies (or growth stocks) and short positions in large companies (or value stocks). Similarly, when stock markets are moving strongly in one direction (i.e., bull or bear), the correlation between individual stocks tends to increase, producing profits when returns generated on the long positions are offset by those generated on short positions. Finally, since small companies and growth stocks tend to be more sensitive to credit and term spreads, long/short equity funds are negatively exposed to the widening of these spreads.

Depending on the different factors, active views on hedge fund returns are determined by the combination of individual factor influences. We assign the value +1 to H (high), the value −1 to L (low), and the value 0 to M, and calculate the sum $S$ of these numbers over all factors. The view on the strategy is bullish if this number is positive (indicating more H occurrences than L occurrences for the strategy at a given date) and bearish if it is negative (indicating more L occurrences than H occurrences for the strategy at a given date). The corresponding value $Q_i$ in the view vector will be defined as: $Q_i = (1 + x \cdot sign(S))\Pi_i$, where $\Pi_i$ is the implied expected return on strategy $i$. This scheme implies that a bullish active bet is obtained (with respect to a neutral estimate) when $S$ is positive, and a bearish view is obtained (with respect to a neutral estimate) when $S$ is negative.

The confidence in our view will be determined by the corresponding $i$-th diagonal value $\Omega_{ii}$ in the variance-covariance matrix of the view distribution, which we define by:

$$\Omega_{ii} = \left(1 - \frac{|S|}{K}\right)\Sigma_{ii}$$

where $K$ is the total number of factors (11 in this case). According to this formulation, it appears that if all factors agree on a bullish or bearish view for a given strategy, then $|S| = K$, and the uncertainty on the view goes to zero, as is normal since confidence is the greatest in that case.

Note also that we use the variance of the returns on the hedge fund strategy $\Sigma_{ii}$ as a multiplicative scaling factor. The intuition is that an investor perceives more uncertainty on a view that relates to a riskier asset, as it is more likely to experience large moves in a given interval of time. Finally, the matrix $P$ is defined so that each line corresponds to an active view and consists of zeros in all but the $i$ place (view on the $i$ asset).

The parameters $x$ and $\tau$ determine the relative weight between neutral portfolio weights and the view-adjusted ones, whose degrees of freedom can be adjusted to generate an active portfolio that will deviate more or less with respect to the benchmark portfolio. More specifically, since the standard deviation of the individual view is perfectly determined by this relation, we use the parameter $\tau$ in order to calibrate the model in terms of the trade-off between neutral and active views. The higher the $\tau$, the lower the relative confidence in the neutral view, and the more weight is put on individual beliefs. Since the parameter $x$ has a function similar to the parameter $\tau$, we fix the former arbitrarily at 40%.

### Implications for Active Style Allocation Decisions

Our opportunity set is composed of the same four hedge fund indexes as before. The sample period is from January 1997 through December 2004. We use the optimal minimum VaR portfolio obtained previously as the reference benchmark portfolio from which we extract neutral implied expected return estimates. To take into account higher moments in hedge fund return distributions, as well as preferences about these higher moments, we apply the four-moment portfolio selection model and use the active views generated by applying the prediction procedure.

We first use Equation (2) to obtain the implied returns ($\Pi$) derived from the "neutral" weights of the minimum value at risk portfolio ($w_N$): $\Pi - R_0 = \alpha_1\beta^{(2)} + \alpha_2\beta^{(3)} + \alpha_3\beta^{(4)}$ where alphas and betas are functions in $w_N$ (see the appendix for more details). Having defined $P$, $Q$, and $\Omega$, we apply Equation (1) in order to obtain the Black-Litterman return vector $E(R_{BL})$. Finally, Equation (2) is used in order to obtain the resulting Black-Litterman portfolio: $E(R_{BL}) - R_0 = \alpha_1\beta^{(2)} + \alpha_2\beta^{(3)} + \alpha_3\beta^{(4)} + \varepsilon$ where alpha and beta parameters are functions of the Black-Litterman weight vector obtained by minimizing the sum of squared residuals ($SSR = \varepsilon'\varepsilon$).

All elements except the parameter $\tau$ have already been determined. Our approach to $\tau$ is to control the tracking error of the portfolio with respect to the benchmark strategic style allocation portfolio. For three different Black-Litterman active portfolios associated with different levels of tracking error, parameter values $\tau$ are 1, 5, and 20.

Exhibit 3 presents summary statistics for the portfolios. The active style selection process, combined with the Black-Litterman portfolio selection method, allows for significant outperformance without a great increase in tracking error, as can be seen from the information ratio values. The excess performance and the tracking error increase with $\tau$, as expected.

## EXHIBIT 3
**Performance of Active Allocation Strategies**

| | minVaR Portfolio | B&L Portfolio $\tau = 1$ | B&L Portfolio $\tau = 5$ | B&L Portfolio $\tau = 20$ |
|---|---|---|---|---|
| Mean annual return | 8.79% | 9.79% | 10.48% | 10.71% |
| Volatility | 4.29% | 4.32% | 4.39% | 4.46% |
| VaR (95%) | 1.33% | 1.21% | 1.17% | 1.19% |
| Sharpe ratio (r = 3%) | 1.58 | 1.80 | 1.93 | 1.95 |
| Tracking error | | 0.86% | 1.24% | 1.37% |
| Information ratio | | 1.17 | 1.37 | 1.41 |

Note at this point that the results presented abstract from a number of issues. First, because we have performed an in-sample factor analysis of hedge fund returns, we do not claim that our process to generate active style allocation views is efficient and should be applied in practice. We believe instead that implementation of dynamic asset allocation strategies requires the use of a more sophisticated econometric process, which should go to great lengths to avoid the pitfalls of data mining.

Nor does our analysis take into account transaction costs. Because of significant shifts in asset allocation (as evidenced in Exhibit 4), we expect such frictions to be a major issue in implementing a style rotation strategy in the hedge fund universe. It is clearly the case that investable pure-strategy hedge fund portfolios used in the process should display a high level of liquidity.

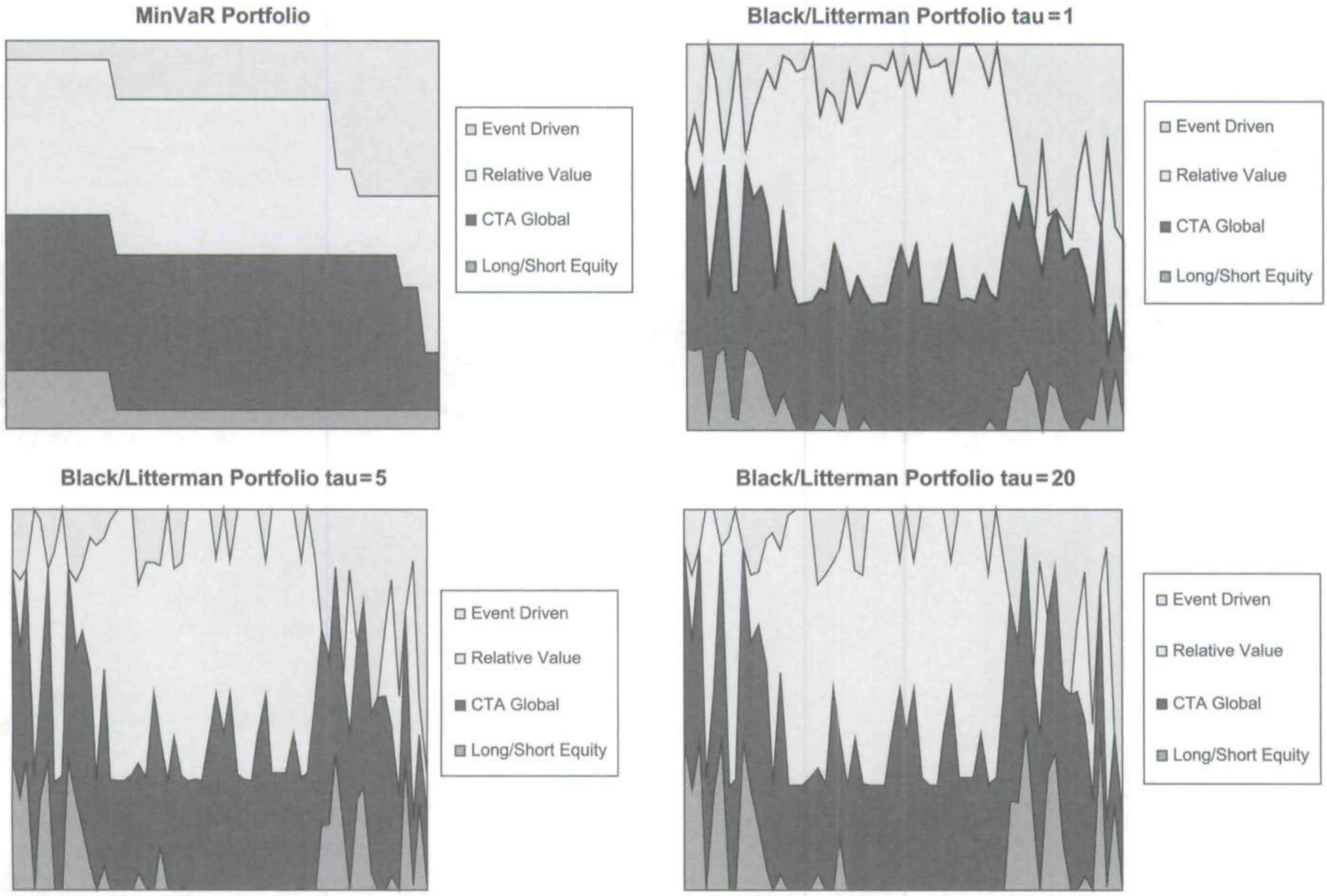
Overall, the purpose of the exercise is not to promote a specific style rotation strategy, but rather to illustrate that, given a set of active views on hedge fund strategy returns, which may come from either a qualitative or a quantitative analysis, optimal active asset allocation decisions can be implemented on the basis of a sound and sophisticated investment process.

## APPENDIX

### Derivation of the Four-Moment CAPM

According to the basics of utility maximization, the mean-variance approach represents only a second-order approximation

## Exhibit 4
**Portfolio Allocations**




of a general utility function in the case of non-normal asset returns. To take higher moments into account, we consider any arbitrary utility function. The investor is assumed to be maximizing the utility emanating from wealth invested in a portfolio with return denoted by $R$.

The fourth-order Taylor expansion is written as:

$$U(R) = \sum_{k=0}^{4} \left[ \frac{U^{(k)}(E(R))}{k!} (R - E(R))^k \right] + o[(R - E(R))^4]$$

where $U^{(k)}$ denotes the $K$-th derivative of the function $U$. Taking the expectation on both sides yields:

$$E[U(R)] \approx U(E(R)) + \frac{U^{(2)}(E(R))}{2} \mu^{(2)}(R) + \frac{U^{(3)}(E(R))}{6} \mu^{(3)}(R) + \frac{U^{(4)}(E(R))}{24} \mu^{(4)}(R)$$

with the centralized moments:

$$\mu^{(2)}(R) = \mathrm{E}[(R - \mathrm{E}(R))^2]$$
$$\mu^{(3)}(R) = \mathrm{E}[(R - \mathrm{E}(R))^3]$$
$$\mu^{(4)}(R) = \mathrm{E}[(R - \mathrm{E}(R))^4]$$

Note that the definitions of higher-order central portfolio moments can be used to express in turn the variance *Var(R)*, skewness *S(R)*, and (excess) kurtosis *K(R)* as follows:

$$Var(R) = \mu^{(2)}(R)$$
$$S(R) = \frac{\mu^{(3)}(R)}{[\mu^{(2)}(R)]^{3/2}}$$
$$K(R) = \frac{\mu^{(4)}(R)}{[\mu^{(2)}(R)]^2} - 3$$

Thus, we can approximate any utility function of a portfolio return as a function of expected portfolio return and standard deviation, but also as a function of the third and fourth moments of the portfolio return distribution.

The new maximization problem yields:

$$\max_{\omega} \Phi(\mu(R_p),\ \mu^{(2)}(R_p),\ \mu^{(3)}(R_p),\ \mu^{(4)}(R_p))$$

such that

$$\Sigma_{i=1}^{n} w_i = 1 - w_0$$

with:

$$\mu(R_P) = E(w'R) + w_0 R_0 = w'E(R) + w_0 R_0$$
$$\mu^{(2)}(R_p) = w'E[(R - E(R))(R - E(R))']w$$
$$= w'\Sigma w$$
$$\mu^{(3)}(R_p) = w'E[(R - E(R))(R - E(R))'$$
$$\otimes (R - E(R))'](w \otimes w) = w'\Omega_w$$
$$\mu^{(4)}(R_p) = w' E[(R - E(R))(R - E(R))'$$
$$\otimes (R - E(R))'$$
$$\otimes (R - E(R))'](w \otimes w \otimes w) = w'\Psi_w$$

where $\otimes$ denotes the Kronecker product; $R = (R_1, ..., R_n)'$ the vector of asset returns; $w = (w_1, ..., w_n)'$ the vector of portfolio proportions invested in these assets; $w_0$ the position held in the risk-free asset (with return $R_0$); and $\mu = (\mu_1, ..., \mu_n)'$ the mean return vector. $\Omega_w$ is the vector of co-skewnesses for the weighting vector $w$ and $\Psi_\omega$ the vector of co-kurtoses. We take this matrix representation from Jondeau and Rockinger [2006], who use it in a different context.

Solving the problem as in Hwang and Satchell [1999], for example, yields Equation (2).

## ENDNOTES

[1]For more details on dynamic allocation decisions with parameter uncertainty, see Detemple [1986], Dothan and Feldman [1986], Gennotte [1986], Brennan [1998], Brennan and Xia [2001], Cvitanic et al. [2005], and Xia [2001], in a continuous-time setting, or Kandel and Stambaugh [1996], Pastor and Stambaugh [1999, 2000], Barberis [2000], and Pastor [2000] in a discrete-time setting.

[2]More generally, this reverse-engineering mechanism can be applied to deduce the expected return estimate that can support *any* given asset allocation, and not necessarily an asset allocation corresponding to market cap weightings.

[3]According to CSFB Tremont, these four broad categories made up 91% of assets under management in the hedge fund industry at the end of 2003. In the CISDM hedge fund database, funds in these strategies constitute 85% of total assets under management by single hedge funds.

[4]For more information, see www.edhec-risk.com, where monthly data on Edhec Alternative Indexes can be downloaded, Return data for the strategy labeled as "CTA Global" is given by the performance of an equally weighted portfolio of Global Macro and CTA indexes.

[5]We use a confidence level of 95%.

[6]Each combination is characterized by a strategy, a factor, and a state of nature. With four strategies, factors, and 3 states of nature, we obtain 132 combinations.

[7]We use a 20% significance level in order to generate a reasonably large number of H and L values.

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